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The scattered spectrum in a homogeneous isotropic turbulent medium

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Abstract. The spectral distribution of electromagnetic waves scattered by density fluctuations in a homogeneous, isotropic turbulent medium is studied. The time-dependent density fluctuations are derived from the hydrodynamic equations, which are quasi-linearised with respect to the acoustic parameters. The interaction between the acoustic field and the vortex field is retained. The scattered power spectrum due to refractive index fluctuations is evaluated for stationary turbulence at high Reynolds number, and the physical spectrum is evaluated for non-stationary turbulence at low Reynolds number. The frequency distribution of the scattered wave is found to be similar to that of a simple fluid. By including the turbulent acoustic effect, this work extends the results of Mountain.

1. Introduction

The spectrum of scattered radiation has long been used to probe the dynamics of matter (Chu 1974, Berne and Pecora 1976, Crosignani *et al* 1974). As early as the 1930s, the Rayleigh line and Brillouin doublets of fluid were predicted (Brillouin 1922) and observed (Gross 1932). Due to the rapid progress of laser techniques, the fine structure of the Rayleigh and Brillouin spectra of fluids can be measured (Benedek 1966). Mountain (1966) linearised the hydrodynamic equations, and successfully explained the fine structure of the Rayleigh and Brillouin spectra of a simple fluid.

It was shown (Moyal 1952) that there are two distinct physical aspects to a compressible turbulent medium. One is called eddy turbulence, which is connected with the breakdown of laminar flow and the creation of a fluctuating eddy motion, that is, the generation of motion having non-zero vorticity. The other is connected with the existence of a fluctuating compression wave, having the character of random noise, the one that Mountain considered in a simple quiet fluid†. There is no interaction between these two effects in the first-order approximation; the nonlinear term, however, will represent the interaction of these two motions. At high levels of turbulence, the eddy turbulence can act as a source of acoustic waves. This phenomenon has been a subject of interest to aerodynamic physicists (Goldstein 1974, Laufer 1974) after Lighthill's (1952) initiation of the theory of acoustic waves generated by turbulence. The primary purpose of our work is to analyse the spectrum of electromagnetic waves scattered by density fluctuations resulting from the effects of these two kinds of motion.

† The quiet fluid is used for emphasising the fact that beside the local thermal noise there is no external noise.

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2. The scattering theory of light

The dielectric constant of a thermally fluctuating medium can be written as

$$\varepsilon(\boldsymbol{r}, t) = \varepsilon_0 + \varepsilon_1(\boldsymbol{r}, t)$$

where ε_0 is the ensemble average value of the dielectric constant of the medium and $\varepsilon_1(\boldsymbol{r}, t)$ is the small fluctuating part, which is changing with respect to space $\boldsymbol{r}$ and time t. If we take the density ρ and temperature T as the thermal state variables, then the fluctuation of the dielectric constant depends on the fluctuation of these two variables. (Just like the dielectric constant, all the thermal variables with subscript zeros will indicate the ensemble averages of the variables; subscript ones will indicate the fluctuating quantities of the variables.) Since the effect due to the latter is less than the former,

$$\mathrm{d}\varepsilon = (\partial\varepsilon/\partial\rho)_T \,\mathrm{d}\rho.$$

The scattered wave intensity due to the density fluctuations can be calculated from Maxwell's equations. According to the definition of running spectrum (Benedek 1966), one can write the instantaneous power spectrum as

$$I(\omega, \boldsymbol{r}, t) = \frac{ck_0^4E_0^2}{(4\pi)(4\pi r)^2}\sin^2\psi(\sqrt{\varepsilon_0})\left(\frac{1}{\varepsilon_0}\frac{\partial\varepsilon}{\partial\rho}\right)^2\int_0^\infty \mathrm{d}\tau\cos\omega\tau$$
$$\times\langle[\Lambda(\boldsymbol{r}, t)+\Lambda^*(\boldsymbol{r}, t)][\Lambda(\boldsymbol{r}, t-\tau)+\Lambda^*(\boldsymbol{r}, t-\tau)]\rangle, \qquad (1)$$

where

$$\Lambda(\boldsymbol{r}, t) = \exp[\mathrm{i}(k_0r-\omega_0t)]\int \mathrm{d}\boldsymbol{r}'\,\rho_1(\boldsymbol{r}, t-|\boldsymbol{r}-\boldsymbol{r}'|/c_\mathrm{m})\exp(\mathrm{i}\boldsymbol{k}_1\cdot\boldsymbol{r}'), \qquad (2)$$

$\langle\ \rangle$ indicates the ensemble average, $\Lambda^*(\boldsymbol{r}, t)$ is the complex conjugate of $\Lambda(\boldsymbol{r}, t)$. $\boldsymbol{k}_0$, $\boldsymbol{E}_0$ are the wave vector and the electric field of the incident wave respectively. The scattered wave vector $\boldsymbol{k}_1 = \boldsymbol{k}_0 - (\omega_0/c_\mathrm{m})\boldsymbol{\eta}$, in which c_m is the velocity of light in the medium, and $\boldsymbol{\eta} = \boldsymbol{r}/|\boldsymbol{r}|$; then ψ is the angle between $\boldsymbol{E}_0$ and $\boldsymbol{\eta}$.

The power spectrum and its Fourier transform, the autocorrelation function, are widely used as a powerful tool for analysing a stationary process, i.e. a time invariant process. If the density fluctuation is stationary, due to ergodic hypothesis, the ensemble average is equal to the time average, hence the autocorrelation function of the density only depends on the time difference of densities. The instantaneous power spectrum accordingly can be reduced to the conventional power spectrum (Wiener–Khintchine theorem), and there is no physical ambiguity in the spectrum due to the fact that the correlation function has already been smoothed out in the time domain. But one does find physical ambiguities in some non-stationary stochastic processes (Bendat and Piersol 1971, Woodward 1953). Mark (1970) introduced a window function to smooth out the ambiguity and called it the physical spectrum.

In this paper, we will evaluate the scattered power spectrum for a stationary turbulent medium, and the physical spectrum for a non-stationary turbulent medium.

3. The hydrodynamical equations and turbulent noise

For a continuous medium, the fluctuations of density ρ and temperature T are determined by the hydrodynamical equations. By use of some thermal properties

(Hunt 1957), one can simplify the Navier–Stokes equation and obtain

$$\partial \boldsymbol{v}/\partial t+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}=-(c_0^2/\gamma)(\nabla \ln\rho+\beta\nabla T)+\nu\nabla^2\boldsymbol{v}+(\xi+\tfrac{1}{3}\eta)\rho^{-1}\nabla(\nabla\cdot\boldsymbol{v}), \tag{3}$$

where c_0 is the speed of sound, γ the ratio of specific heat, β the coefficient of thermal expansion, ξ the shear viscosity, η the bulk viscosity, and ν the dynamic viscosity. Assuming all these thermal coefficients are constant and taking the divergence of (3), we have

$$\partial\phi/\partial t+(\boldsymbol{v}\cdot\nabla)\phi-b\nabla^2\phi+(c_0^2/\gamma)(\nabla^2 X+\beta\nabla^2 T)=-(\partial v_i/\partial x_j)\,\partial v_j/\partial x_i, \tag{4}$$

where $b=(\xi+\frac{4}{3}\eta)/\rho_0$, $\phi=\nabla\cdot\boldsymbol{v}$, and $X=\ln(\rho)$. In a homogeneous turbulent medium, the thermal quantities X, T, and velocity $\boldsymbol{v}$ are all fluctuating about their mean values, which are independent of space and time. The fluctuation of the velocity can be separated into two parts, the irrotational part expressed as $\boldsymbol{u}^{\mathrm{p}}$ (i.e. $\nabla\times\boldsymbol{u}^{\mathrm{p}}=0$), and the rotational part expressed as $\boldsymbol{u}^{\mathrm{s}}$ (i.e. $\nabla\cdot\boldsymbol{u}^{\mathrm{s}}=0$). Since $|\boldsymbol{u}^{\mathrm{p}}|\simeq(U/c_0)^2|\boldsymbol{u}^{\mathrm{s}}|$ (Monin and Yaglom 1971) where U is the average speed of flow, in a subsonic flow $|\boldsymbol{u}^{\mathrm{p}}|\ll|\boldsymbol{u}^{\mathrm{s}}|$. Taking the lowest order of $(\partial v_i/\partial x_j)\,\partial v_j/\partial x_i$, and neglecting the viscous dissipation of the medium (Hinze 1959), we can write down the quasi-linearised hydrodynamic equations as

$$\begin{gathered}\partial X_1/\partial t+(\boldsymbol{U}\cdot\nabla)X_1+\phi_1=0,\\ \frac{\partial\phi_1}{\partial t}+(\boldsymbol{U}\cdot\nabla)\phi_1-b\nabla^2\phi_1+\frac{c_0^2}{\gamma}(\nabla^2X_1+\beta\nabla^2T_1)=-\frac{\partial u_i^{\mathrm{s}}}{\partial x_j}\frac{\partial u_j^{\mathrm{s}}}{\partial x_i},\\ \partial T_1/\partial t+(\boldsymbol{U}\cdot\nabla)T_1+[(\gamma-1)/\beta]\phi_1-\chi\nabla^2T_1=0,\end{gathered} \tag{5}$$

where $\chi=\kappa/\rho_0c_v$ (κ is the thermal conductivity of the medium, and c_v is the specific heat at constant volume of the medium). For a homogeneous isotropic turbulent medium, we have to take $U=0$, which can be achieved by moving the observer with the flow. Let

$$a_1(\boldsymbol{r},t)=(8\pi^3)^{-1/2}\int \mathrm{d}\boldsymbol{k}\,\hat{a}_1(\boldsymbol{k},t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$$

indicate the Fourier transform relations of X_1, T_1 and ϕ_1. By Fourier transforming (5), we have

$$\begin{gathered}\partial\hat{X}_1/\partial t+\hat{\phi}_1=0,\\ \partial\hat{\phi}_1/\partial t+bk^2\hat{\phi}_1-(c_0^2k^2/\gamma)(\hat{X}_1+\beta\hat{T}_1)=-\hat{F},\\ \partial\hat{T}_1/\partial t+[(\gamma-1)/\beta]\hat{\phi}_1+\chi k^2\hat{T}_1=0,\end{gathered} \tag{6}$$

where

$$\hat{F}=(8\pi^3)^{-1/2}\int \mathrm{d}\boldsymbol{x}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_i^{\mathrm{s}}u_j^{\mathrm{s}}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}.$$

The solution of this set of non-homogeneous differential equations contains the homogeneous solution and special solution. The homogeneous solution has been obtained by Mountain (1966), who used the solution to explain the scattered spectrum of a simple fluid. The purpose of this paper is to find the special solution of (6) and treat it as a medium with external noise, which is produced by the turbulence of the medium. This special solution can be obtained by introducing a set of Green functions. Let

$$G_i(\boldsymbol{k}, t, t') = \int \mathrm{d}\omega\, g_i(\boldsymbol{k}, \omega) \exp[-\mathrm{i}\omega(t-t')] \tag{7}$$

be the Green functions of $\hat{X}_1$, $\hat{\phi}_1$ and $\hat{T}_1$. Substituting the relations of (7) into (6), we have

$$\begin{bmatrix} \mathrm{i}\omega & -1 & 0 \\ \dfrac{c_0^2 k^2}{\gamma} & \mathrm{i}\omega - bk^2 & \dfrac{c_0^2 k^2}{\gamma}\beta \\ 0 & -\dfrac{\gamma-1}{\beta} & \mathrm{i}\omega - \chi k^2 \end{bmatrix} \begin{bmatrix} g_X \\ g_\phi \\ g_T \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{1}{2\pi} \\ 0 \end{bmatrix}. \tag{8}$$

From solving (8), we have

$$g_X = \left(\frac{1}{2\pi}\right)\left(\frac{\mathrm{i}\omega - \chi k^2}{(\mathrm{i}\omega)^3 - (bk^2 + \chi k^2)(\mathrm{i}\omega)^2 + (\chi b k^4 + c_0^2 k^2)(\mathrm{i}\omega) - (c_0^2 k^2/\gamma)\chi k^2}\right).$$

Let $\Gamma = \frac{1}{2}[b + (1 - 1/\gamma)]$; then $\chi k^2/c_0 k$ and $\Gamma k^2/c_0 k$ will be of the order of $\omega_{\mathrm{th}}/\omega_0$, where ω_{th} is the frequency of thermal variation and ω_0 is the frequency of the incident wave, which is always less than one for a continuous medium; hence the first-order approximation of the denominator of g_X is

$$M \simeq (\mathrm{i}\omega - \chi k^2/\gamma)(\mathrm{i}\omega + \mathrm{i}c_0 k - \Gamma k^2)(\mathrm{i}\omega - \mathrm{i}c_0 k - \Gamma k^2),$$

and one finds

$$\begin{aligned} G_X(\boldsymbol{k}, t, t') &\simeq \frac{\chi k^2}{c_0 k}\left(1 - \frac{1}{\gamma}\right)\frac{1}{c_0 k}\left[\exp\left(\frac{-\chi k^2}{\gamma}(t-t')\right) - \exp[-\Gamma k^2(t-t')]\cos c_0 k(t-t')\right] \\ &\quad + (c_0 k)^{-1}\exp[-\Gamma k^2(t-t')]\sin c_0 k(t-t'), \qquad \text{for } t \geqslant t', \\ G_X(\boldsymbol{k}, t, t') &= 0, \qquad \text{for } t < t'. \end{aligned} \tag{9}$$

Since the density fluctuation is affected by both the local thermal agitation and the turbulence of the medium, we can write the density fluctuations as

$$\hat{\rho}_1 = \hat{\rho}_{\mathrm{H}} + \hat{\rho}_{\mathrm{s}} \tag{10}$$

where $\hat{\rho}_{\mathrm{H}}$ is the density fluctuation due to the local thermal agitation, which is the homogeneous solution of (6), and $\hat{\rho}_{\mathrm{s}}$ is the density fluctuation due to the turbulent noise of the medium, which is the special solution of (6). By using the Green function $G_X(\boldsymbol{k}, t, t')$, we can write

$$\hat{\rho}_{\mathrm{s}} \simeq \rho_0 \int_{-\infty}^{\infty} \mathrm{d}t\, G_X(\boldsymbol{k}, t, t')\hat{F}(\boldsymbol{k}, t'). \tag{11}$$

The instantaneous fluctuating quantities usually are not measurable, so it is more practical just to find the correlation function or the structure function of the fluctuating quantities. However, we are only interested in the correlation function of densities. If we neglect the interaction between local thermal agitation and turbulent effect, we have

$$\langle \hat{\rho}_1(\boldsymbol{k}, t_1)\hat{\rho}_1(\boldsymbol{k}, t_2)\rangle = \langle \hat{\rho}_{\mathrm{H}}(\boldsymbol{k}, t_1)\hat{\rho}_{\mathrm{H}}(\boldsymbol{k}, t_2)\rangle + \langle \hat{\rho}_{\mathrm{s}}(\boldsymbol{k}, t_1)\hat{\rho}_{\mathrm{s}}(\boldsymbol{k}, t_2)\rangle. \tag{12}$$

The first part on the right-hand side of (12), caused by the local thermal agitation of the medium, has been evaluated by Mountain (1966); the second part, caused by the turbulence of the medium, can be evaluated by substituting (11) into (12). According to the last statement, we have

$$\langle \hat{\rho}_s(\boldsymbol{k}, t_1)\hat{\rho}_s(\boldsymbol{k}, t_2)\rangle = \rho_0^2 \int dt' \int dt''\, G_X(\boldsymbol{k}, t_1, t')G_X(\boldsymbol{k}, t_2, t'')\langle \hat{F}(\boldsymbol{k}, t')\hat{F}(\boldsymbol{k}, t'')\rangle. \tag{13}$$

From (13) and the definition of $\hat{F}(\boldsymbol{k}, t)$, we can see that the density correlation function is determined by the quadrupole correlation of velocities. As the correlation function is very difficult to deal with, for simplicity, an isotropic homogeneous turbulent medium is assumed. This type of turbulence can be generated downstream from a regular array of rods. The velocity distribution in a homogeneous isotropic turbulent medium is approximately normal (Batchelor 1960), so one can decompose the quadrupole velocity correlation function to be the combination of double velocity correlation functions. By knowing the double correlation function of velocities, one can calculate the density correlation function and obtain the scattered spectrum.

4. Turbulent noise power spectrum

For evaluating the scattered spectrum due to turbulent effect, we write

$$\Lambda(\boldsymbol{r}, t) = \Lambda_H(\boldsymbol{r}, t) + \Lambda_s(\boldsymbol{r}, t),$$

where Λ_H indicates the local thermal effect, Λ_s the turbulent effect. Substituting (11) into the definition of $\Lambda(\boldsymbol{r}, t)$, equation (2), we can write

$$\Lambda_s(\boldsymbol{r}, t) = \rho_0 \exp[i(k_0 r - \omega_0 t)] \int dt' \int d\boldsymbol{x}' \exp(i\boldsymbol{k}_1 \cdot \boldsymbol{x}) k_1^i k_1^j u_i^s u_j^s G_X(\boldsymbol{k}_1, t - r/c_m, t'). \tag{14}$$

For avoiding the physical ambiguity in a non-stationary turbulent medium, we will evaluate the physical scattered spectrum for a non-stationary turbulent medium. In a stationary turbulent medium, there is no difference between the conventional power spectrum and the physical spectrum (Mark 1970); we will only evaluate the conventional power spectrum for the stationary case.

4.1. Non-stationary turbulence and physical spectrum

For a non-stationary stochastic process, a window function has to be introduced in order to have a practical power spectrum. Let $w(t-\mu)$ be the window function, which is positive in the neighbourhood of $t-\mu=0$, and small outside this neighbourhood. If we have

$$W(\omega, \boldsymbol{r}, t) = \int_{-\infty}^{\infty} d\mu\, w(t-\mu)[\Lambda_s(\boldsymbol{r}, \mu) + \Lambda_s^*(\boldsymbol{r}, \mu)]\, e^{i\omega\mu}, \tag{15}$$

where $\Lambda_s(\boldsymbol{r}, t)$ has been defined in (2), then the physical power spectrum can be defined as $\langle |W(\omega, \boldsymbol{r}, t)|^2\rangle$. Thus the scattered physical power spectrum of a non-stationary turbulent medium can be written as

$$\langle |W(\omega, \boldsymbol{r}, t)|^2\rangle = \rho_0^2 \int d\mu \int d\mu' \int dt' \int dt'' \int d\boldsymbol{x}' \int d\boldsymbol{x}''$$
$$\times W(t-\mu) W(t-\mu') G_X(\boldsymbol{k}_1, t-\boldsymbol{r}/c_m, t') G_X(\boldsymbol{k}, t-\boldsymbol{r}/c_m, t'')$$
$$\times i^4 k_1^i k_1^j k_1^l k_1^m \langle u_i^s(\boldsymbol{x}', t') u_j^s(\boldsymbol{x}', t') u_l^s(\boldsymbol{x}'', t'') u_m^s(\boldsymbol{x}'', t'')\rangle$$
$$\times \{\exp[-i\omega_0(\mu-\mu')] \exp[i\boldsymbol{k}_1 \cdot (\boldsymbol{x}'-\boldsymbol{x}'')] + cc$$
$$+ \exp(2i\boldsymbol{k}_1 \cdot \boldsymbol{r}) \exp[-i\omega_0(\mu+\mu')] + cc\}. \tag{16}$$

The window function is taken as

$$W(t-\mu) = \begin{cases} (2\alpha)^{-1/2} & |t-\mu| < \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where α is the time width for short time averaging. The quadrupole correlation of velocities is evaluated for a homogeneous isotropic turbulent medium. The fluctuation of velocity is assumed to be normally distributed, so one can decompose the quadrupole correlation function into the combination of double correlation functions, that is

$$\langle u_i u_j u_l' u_m'\rangle = \langle u_i u_j\rangle\langle u_l' u_m'\rangle + \langle u_i u_l'\rangle\langle u_j u_m'\rangle + \langle u_i u_m'\rangle\langle u_j u_l'\rangle. \tag{17}$$

Let

$$Q_{ij}(\boldsymbol{\xi}, t, t+\Delta t) = \langle u_i(\boldsymbol{r}, t) u_j(\boldsymbol{r}', t')\rangle,$$

where $\boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{r}'$ and $\Delta t = t' - t$; $E_{ij}(\boldsymbol{k}, t, t+\Delta t)$ is the Fourier transform of Q_{ij}. The physical power spectrum can be evaluated by knowing that the energy spectrum tensor (Deissler 1960),

$$E_{ij}(\boldsymbol{k}, t', t'') = \phi(k, t', t'')(k^2\delta_{ij} - k_i k_j),$$
$$\phi(k, t', t'') = c_k \exp(-2k^2/k_m^2) \exp[-\nu k^2(t' + t'' - 2t_0)], \tag{18}$$

where t_0 is the initial time. For simplicity, we take

$$G_X(k, t, t') = \begin{cases} (c_0 k)^{-1} \exp[-\Gamma k^2(t-t')] \sin c_0 k(t-t'), & t \geqslant t', \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

in place of (9) since both $(1-1/\gamma)$ and $\chi k^2/c_0 k$ are smaller than 1.

4.2. *Stationary turbulence and power spectrum*

The conventional power spectrum (Wiener–Khintchine theorem) is taken for evaluating the scattered spectrum of stationary turbulence. Substituting (14) into (1), we have

$$I_s(\omega, \boldsymbol{r}, t) = (\text{factor}) \int_{-\infty}^{\infty} d\tau \cos\omega\tau \int dt' \int dt'' \int d\boldsymbol{x}' \int d\boldsymbol{x}''$$
$$\times G_X(\boldsymbol{k}_1, t, t') G_X(\boldsymbol{k}_1, t-\tau, t'') i^4 k_1^i k_1^j k_1^l k_1^m$$
$$\times \langle u_i(\boldsymbol{x}', t') u_j(\boldsymbol{x}', t') u_l(\boldsymbol{x}'', t'') u_m(\boldsymbol{x}'', t'')\rangle$$
$$\times \{\exp(-i\omega_0\tau) \exp[i\boldsymbol{k}_0 \cdot (\boldsymbol{x}'-\boldsymbol{x}'')] + cc$$
$$+ \exp(2ik_0 r) \exp[-i\omega_0(2t-\tau)] + cc\}. \tag{20}$$

A homogeneous isotropic turbulence is also assumed, and the velocity fluctuations are assumed to be normally distributed. But the energy spectrum of stationary turbulence is taken as (Kraichnan 1976)

$$E_{ij}(\boldsymbol{k}, t, t+\Delta t) = (\text{constant})(K^2\delta_{ij} - k_ik_j)\varepsilon^{2/3}k^{-17/3}\exp[-\tfrac{1}{2}k^2v_l^2(\Delta t)^2], \tag{21}$$

where v_l is the characteristic velocity of large eddies. The energy spectrum is taken in the inertial sub-range, from k_l to k_d, which correspond to the largest and smallest scales of the eddies.

5. The differential cross section of the scattered wave

In a turbulent medium, the density fluctuations contain not only the information of local thermal agitation, but also the information of turbulence. Without being in the region of turbulence, we hope to evaluate the turbulent noise effect in the scattered spectrum in order to predict the structure of turbulence.

The differential cross section of the scattered wave is defined as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{I(\omega, \boldsymbol{r}, t)r^2}{I_0}, \qquad I_0 = \frac{c}{4\pi}E_0^2\varepsilon^{2/3},$$

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\omega\,\mathrm{d}\Omega} = {}& \frac{k_0^4\sin^2\psi}{(4\pi)^2}\left(\frac{\partial\varepsilon}{\partial\rho}\frac{\rho_0}{\varepsilon_0}\right)^2\tfrac{1}{2}V_{sc}4\pi \\ & \times\left\{\frac{(\chi k_1^2/\gamma)^2}{(\chi k_1^2/\gamma)^2 + (\omega-\omega_0)^2}(\gamma-1)k_1^{-3}\left(\frac{v_l}{c_0}\right)^3\left(\frac{1}{c_0k_1}\right)\left(\frac{k_l}{k_1}\right)^{4/3}\right. \\ & + \left[\frac{(\Gamma k_1^2)^2}{(\Gamma k_1^2)^2 + (\omega-\omega_0-c_0k_1)^2} + \frac{(\Gamma k_1^2)^2}{(\Gamma k_1^2)^2 + (\omega-\omega_0+c_0k_1)^2}\right]\frac{1}{4} \\ & \left.\times k_d^{-3}\left(\frac{v_l}{c_0}\right)^3\left(\frac{k_l}{k_d}\right)^{4/3}\left(\frac{k_1}{k_d}\right)\left(\frac{1}{\Gamma k_1^2}\right)\left(\frac{c_0k_1}{\Gamma k_1^2}\right)\exp\left[-\frac{1}{2}\left(\frac{c_0}{v_l}\right)^2\left(\frac{k_l}{k_d}\right)^2\right]\right\},\end{aligned}$$

for the stationary fluctuation, and

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\omega\,\mathrm{d}\Omega} = {}& \frac{k_0^4\sin^2\psi}{(4\pi)^2}\left(\frac{\partial\varepsilon\rho_0}{\partial\rho\varepsilon_0}\right)^2\frac{\alpha}{2}\left(\frac{q}{c_0}\right)^4(16\times\tfrac{8}{3})k_m^{-3}V_{sc} \\ & \times\left\{\frac{(\nu k_1^2)^2}{(\nu k_1^2)^2 + (\omega-\omega_0)^2}\left[2\nu k_1^2\left(t - t_0 - \frac{r}{c_m} + \frac{1}{\nu k_m^2}\right)\right]^{-7/2}\right. \\ & \times\exp\left[-\nu k_1^2\left(t - t_0 - \frac{r}{c_m} + \frac{1}{\nu k_m^2}\right)\right] \\ & + \left[\frac{(\Gamma k_1^2)^2}{(\Gamma k_1^2)^2 + (\omega-\omega_0+c_0k_1)^2} + \frac{(\Gamma k_1^2)^2}{(\Gamma k_1^2)^2 + (\omega-\omega_0-c_0k_1)^2}\right]\frac{1}{4}2^{-7/2} \\ & \left.\times\exp\left[-\Gamma k_1^2\left(t - t_0 - \frac{r}{c_m}\right)\right]\exp\left(\frac{-k_1^2}{k_m^2}\right)\right\},\end{aligned}$$

for the non-stationary fluctuation, but note the width is $O(1/\alpha)$ instead of (νk_1^2) or (Γk_1^2).

The initial condition of the energy spectrum tensor in (18) is taken as (Kraichnan 1964)

$$\phi(k,0)=(16)(2/\pi)V_0^2k_m^{-5}k^4\exp(-k^2/k_m^2),$$

where k_m is the wavenumber at which the energy is maximum. An isotropic homogeneous turbulence was produced by Helland and Van Atta (1977) in a wind tunnel. By matching the energy spectrum tensor with the experiment (figure 1), we find that k_m is of the order of $10^{-1}\,cm^{-1}$. Even the energy spectrum tensor of Kraichnan's model does not match with Helland's experiment very well; for theoretical reasons we adopt the data (table 1) from the experiment and evaluate the differential cross section for stationary and non-stationary turbulence (table 2).

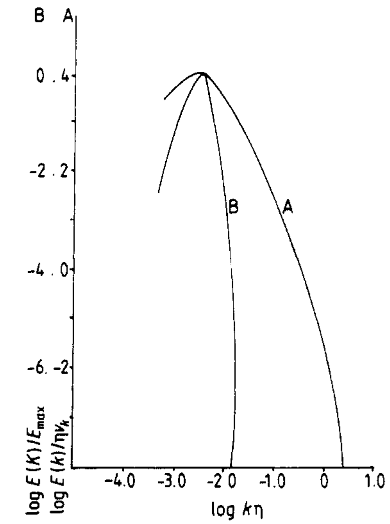

Figure 1. Comparison of three-dimensional energy spectrum. A, Helland and Van Atta grid experiment; B, Kraichnan's work.

Table 1. Parameters for calculation on scattering intensity.

	Air (non-stationary)	Water (stationary)
$k_0\,(cm^{-1})$	10^{-1}	10^2k_l
$\{\rho_0(\partial\varepsilon/\partial\rho)\varepsilon_0^{-1}\}$	6×10^{-4}	0.82
$k_l\,(cm^{-1})$	0.2	10^3
$k_d\,(cm^{-1})$	3.15×10^2	10^3k_l
c (cm/s)	10^4	10^5
γ	1.4	1.004
$k_m\,(cm^{-1})$	0.1	
$V_{sc}\,(cm^3)$	10^3	
K_T	10^{-6}	10^{-12}
T (°K)	300	
$q^2\,(cm^2/s^2)$	100	100
α (s)		10^{-2}

Table 2. The scattered differential cross section.

	Stationary	Non-stationary
$(\mathrm{d}\sigma/\mathrm{d}\omega\ \mathrm{d}\Omega)_{\text{turbulence only}}$	$O(10^{-2})$	$O(10^{-26})$
$(\mathrm{d}\sigma/\mathrm{d}\omega\ \mathrm{d}\Omega)_{\text{quiet}}$	$O(10^{-6})$	$O(10^{-30})$
$\mathcal{I}_c$	$O(10^4)$	$O(10^4)$
$2I_B/I_c$	e^{-25}	Time dependent

6. Conclusion and discussion

The turbulent noise effect on the scattered spectrum should be observed outside the region of turbulence, or the local turbulent effect will be far more distinct than the turbulent noise effect. In an isotropic homogeneous turbulent medium, the scattered spectrum can only be measured inside the region of turbulence, so the central maximum is very large compared with the Brillouin doublet. Obviously a more practical system is needed in order to evaluate the turbulent noise effect on the scattered spectrum, such as a turbulent jet. For theoretical reasons, we did preliminary work in a homogeneous isotropic turbulent medium. By comparing the scattered spectrum of a quiet simple fluid (Berne and Pecora 1976), we find

$\mathcal{I}_c$ = the scattered intensity ratio at central maximum of an isotropic homogeneous medium to the quiet simple fluid

$$= \frac{(\chi k_1^2/c_0 k_1)\gamma(v_l/c_0)^3 4(k_l/k_0)^4_3 k_1^{-3}}{K_T K_B T}$$

for stationary turbulence, and

$$\mathcal{I}_c = \frac{(q/c)^4 k_m^{-3} 4}{(1-1/\gamma)K_B K_T T}(2)^{-7/2} \ell^{-7/2}\, e^{-\ell}$$

for non-stationary turbulence, where K_T is the isothermal compressibility of the medium, K_B is the Boltzmann constant and $\ell = \nu k_1^2(t - t_0 + 1/\nu k_m^2 - r/c_m)$. The scattered intensity of non-stationary turbulence depends on time; we assume that the time interval is less than $1/\nu k_1^2$ and $r/c_m \ll 1/\nu k_m^2$. Since the Brillouin doublet is far less than the central maximum, we only estimate the order of magnitude of the scattered intensity at the central maximum. Water and air are taken as the scattered media for stationary and non-stationary turbulence respectively.

In table 2, we find that $\mathcal{I}_c$ is of the order of 10^4 for both stationary and non-stationary turbulence. But the scattered intensity for air is only of the order of 10^{-30}, so the scattered spectrum in air is hardly detectable. We expect to measure the scattered spectrum in a dense medium.

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